

# Exact matrix product solution for the boundary-driven Lindblad $XXZ$ -chain

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We demonstrate that the exact non-equilibrium steady state of the one-dimensional Heisenberg  $XXZ$  spin chain driven by boundary Lindblad operators can be constructed explicitly with a matrix product ansatz for the non-equilibrium density matrix where the matrices satisfy a *quadratic algebra*. This algebra turns out to be related to the quantum algebra  $U_q[SU(2)]$ . Coherent state techniques are introduced for the exact solution of the isotropic Heisenberg chain with and without quantum boundary fields and Lindblad terms that correspond to two different completely polarized boundary states. We show that this boundary twist leads to non-vanishing stationary currents of all spin components. Our results suggest that the matrix product ansatz can be extended to more general quantum systems kept far from equilibrium by Lindblad boundary terms.

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The non-equilibrium behaviour of open quantum systems has become accessible through recent advances in artificially assembled nanomagnets consisting of just a few atoms [1] or in the study of quasi one-dimensional spin chain materials like  $\text{SrCuO}_2$  where many transport characteristics are measurable experimentally [2, 3]. In particular, it is desirable to understand the interplay between many-body *bulk* properties (e.g. magnon excitations or magnetization currents in quantum spin systems) and *local* pumping (applied to the boundary of a system) driving the system constantly out of equilibrium. A good starting point is provided by the anisotropic Heisenberg model [4]

$$H = J \sum_k (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z - \varepsilon_0) \quad (1)$$

of coupled spins. The pure quantum version of this model is exactly solvable by the Bethe ansatz. Interestingly, within linear response theory, i.e., close to equilibrium, it was found that at finite temperature a diffusive contribution to the Drude weight appears [5–7], which is at variance with the long-held belief that integrability protects the ballistic nature of transport phenomena. Unfortunately the Bethe-ansatz fails in the more relevant context of open far-from-equilibrium systems where these questions can be addressed directly in terms of the Lindblad Master equation [8]

$$\frac{d}{dt} \rho = -i [H, \rho] + \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho) \quad (2)$$

for the reduced density matrix  $\rho$  associated to the chain (here and below we set  $\hbar = 1$ ). The dissipative terms  $\mathcal{D}^{L,R}(\rho) = D^{L,R} \rho D^{L,R\dagger} - \frac{1}{2} \left\{ \rho, D^{L,R} D^{L,R\dagger} \right\}$  with the Lindblad operators  $D^{L,R}$  acting locally at the open ends of the quantum chain (see below) describe the coupling to external reservoirs that drive a current through the system and thus keep the system in a permanent non-equilibrium steady state. Indeed, using dissipative dynamics for the preparation of quantum states is becoming a promising field of research [9, 10].

Significant progress has been achieved very recently in two remarkable papers by Prosen [11, 12] who observed that the exact stationary density matrix for the  $XXZ$  chain with one specific pair of Lindblad boundary terms can be constructed explicitly in matrix product operator form [13] by a matrix product ansatz (MPA) somewhat reminiscent of the matrix product ansatz of Derrida et al. [14] for the stationary distribution of purely classical stochastic dynamics. With an explicit representation of the matrix algebra Prosen was then able to compute analytically various physical quantities of interest. However, in contrast to [14], where the matrices satisfy a quadratic algebra, the matrices of [11, 12] satisfy a cubic algebra which arises from a peculiar local cancellation mechanism involving three neighboring sites in the quantum chain. This feature is significant since, due to the lack of a general representation theory for cubic algebras, this approach does not lend itself easily to generalization to other open quantum systems with other cubic algebras or even small modifications of the original problem such as boundary fields or other Lindblad terms for the  $XXZ$  chain which would require a different representation. Indeed, the wide applicability of the MPA of [14] derives from the fact that many quadratic algebras (which include all Lie algebras through their commutation relations) have explicitly known representations which is crucial for the exact computation of physical observables [15].

In this Letter we show that exact non-equilibrium steady states for open quantum systems can be obtained from a matrix product ansatz which yields a *quadratic algebra*. Specifically, we consider the Lindblad quantum  $XXZ$  chain and show that the associated matrix algebra is related to the bulk symmetry of the  $XXZ$ -chain, which is the quantum algebra  $U_q[SU(2)]$  with  $\Delta = (q + q^{-1})/2$ . A coherent state representation makes it possible to consider Lindblad operators that correspond to two different completely polarized boundary states, viz., in the  $(y, z)$  plane on the left boundary

$$D^L = \sqrt{\frac{\Gamma}{2}} (\sigma_1^x + i \cos \theta_L \sigma_1^y - i \sin \theta_L \sigma_1^z) \quad (3)$$

and in the  $(x, z)$  plane on the right boundary with

$$D^R = \sqrt{\frac{\Gamma}{2}}(\cos \theta_R \sigma_N^x - i \sigma_N^y + \sin \theta_R \sigma_N^z). \quad (4)$$

These Lindblad generators lead to local dissipative terms whose stationary solutions, satisfying  $\mathcal{D}^{L(R)}(\rho^{L(R)}) = 0$ , are respectively the pure states  $\rho^L = 1/2(\mathbb{1} + \sigma_u^z) = |\uparrow_u\rangle\langle\uparrow_u|$  and  $\rho^R = 1/2(\mathbb{1} - \sigma_v^z) = |\downarrow_v\rangle\langle\downarrow_v|$ , where  $|\uparrow_u\rangle$  is the eigenstate associated to the eigenvalue  $+1$  of  $\sigma_u^z = \sin \theta_L \sigma^y + \cos \theta_L \sigma^z$ , and  $|\downarrow_v\rangle$  is the eigenstate of  $\sigma_v^z = -\sin \theta_R \sigma^x + \cos \theta_R \sigma^z$  with eigenvalue  $-1$ . For computational convenience we have chosen equal left and right amplitudes  $\Gamma$  in (3) and (4). By a judiciously chosen similarity transformation these amplitudes can be made different [16].

Moreover, we allow for quantum boundary fields acting on the directions of the local polarizations specified by (3) and (4). Consequently we add to the Hamiltonian (1) the contribution  $\tilde{f}^L \cdot \vec{\sigma} = f^L \sigma_u^z$  for the left-end of the chain and  $\tilde{f}^R \cdot \vec{\sigma} = f^R \sigma_v^z$  coming from the right-end boundary field. For convenience we choose  $J = 1/2$  and  $\varepsilon = 1$  so that  $H = \sum_{k=1}^{N-1} h_{k,k+1} + g_1^L + g_N^R$  with the four-by-four matrix  $h = \frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta(\sigma^z \otimes \sigma^z - 1))$  for the nearest neighbour bulk interaction and the two-by-two matrices  $g^L = f^L \sigma_u^z$  and  $g^R = f^R \sigma_v^z$  for the boundary fields. The subscript indicates on which sites of the chain the quantum operators  $g$  and  $h$  act non-trivially. We write the stationary density matrix satisfying

$$i[H, \rho] = \mathcal{D}^L(\rho) + \mathcal{D}^R(\rho) \quad (5)$$

in the standard form  $\rho = SS^\dagger / \text{Tr}(SS^\dagger)$ .

Our starting point for solving (5) is a matrix product ansatz

$$S = \langle \phi | \Omega^{\otimes N} | \psi \rangle \quad (6)$$

which we augment by auxiliary matrices  $\Xi$  such that the *local divergence condition*

$$[h, \Omega \otimes \Omega] = \Xi \otimes \Omega - \Omega \otimes \Xi \quad (7)$$

is satisfied. In this construction

$$\Omega = \begin{pmatrix} A_1 & A_+ \\ A_- & A_2 \end{pmatrix}, \quad \Xi = \begin{pmatrix} E_1 & E_+ \\ E_- & E_2 \end{pmatrix}. \quad (8)$$

are two-by-two matrices whose entries  $A_a, E_a$  are non-commuting matrices that act in a space  $\mathcal{A}$  with inner product  $\langle \cdot | \cdot \rangle$ , and  $\langle \phi |, | \psi \rangle$  are vectors in  $\mathcal{A}$ . In terms of Pauli matrices  $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ ,  $\sigma^z$ , and the two-dimensional unit matrix  $\mathbb{1}$  one can conveniently write  $\Omega = A_0 \mathbb{1} + \vec{A} \cdot \vec{\sigma}$  with  $A_0 = \frac{1}{2}(A_1 + A_2)$ ,  $A_z = \frac{1}{2}(A_1 - A_2)$ ,  $A_x = \frac{1}{2}(A_+ + A_-)$ ,  $A_y = \frac{i}{2}(A_+ - A_-)$ . In our construction the local divergence condition leads to a set of 16 *quadratic* relations for the eight matrices  $A_a, E_a$  and the problem to be attacked is the construction of matrices which satisfy these relations.

Remarkably, all 16 equations (7) can be solved in terms of only three independent matrices  $A_\pm, Q$  with  $QQ^{-1} =$

$Q^{-1}Q = 1$  by choosing the auxiliary matrices  $E_\pm = 0$ ,  $E_1 = (q - q^{-1})(bQ - cQ^{-1})/2$ ,  $E_2 = -(q - q^{-1})(\bar{b}Q - \bar{c}Q^{-1})/2$ , setting

$$A_1 = bQ + cQ^{-1}, \quad A_2 = \bar{b}Q + \bar{c}Q^{-1} \quad (9)$$

and requiring

$$[A_+, A_-] = -(q - q^{-1})(b\bar{b}Q^2 - c\bar{c}Q^{-2}) \quad (10)$$

$$QA_\pm = q^{\pm 1}A_\pm Q. \quad (11)$$

The constants  $b, c, \bar{b}, \bar{c}$  are arbitrary. Choosing the parametrization

$$b = \frac{\alpha}{q - q^{-1}} \frac{\nu}{\lambda}, \quad \bar{b} = \frac{\alpha}{q - q^{-1}} \frac{1}{\lambda \nu} \quad (12)$$

$$c = -\frac{\alpha}{q - q^{-1}} \mu \lambda, \quad \bar{c} = -\frac{\alpha}{q - q^{-1}} \frac{\lambda}{\mu}, \quad (13)$$

and defining

$$A_\pm =: i\alpha S_\pm, \quad Q =: \lambda q^{S_z} \quad (14)$$

then leads to

$$[S_+, S_-] = \frac{q^{2S_z} - q^{-2S_z}}{q - q^{-1}} \quad (15)$$

$$q^{S_z} S_\pm = q^{\pm 1} S_\pm q^{S_z}. \quad (16)$$

These are the defining relations of  $U_q[SU(2)]$ , the  $q$ -deformed universal enveloping algebra of the Lie algebra  $SU(2)$ , which is the non-abelian symmetry of the bulk Hamiltonian (1) [17].

After deriving a matrix algebra from the bulk interactions the second step is the explicit construction of such matrices and of the vectors  $\langle V |$  and  $| W \rangle$  using the boundary interactions. For the present case we note that the representation theory of  $U_q[SU(2)]$  is well-understood and analogous to that of  $SU(2)$ , except when  $q$  is a root of unity, where some special features arise [17]. In particular, with the definition  $[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}$  we have the irreducible representation (irrep)

$$\begin{aligned} S_z &= \sum_{k=0}^{\infty} (p - k) |k\rangle \langle k| \\ S_+ &= \sum_{k=0}^{\infty} [k+1]_q |k+1\rangle \langle k| \\ S_- &= \sum_{k=0}^{\infty} [2p - k]_q |k+1\rangle \langle k| \end{aligned}$$

where  $p$  is an arbitrary complex parameter. This irrep is infinite-dimensional, except when  $2p \in \mathbb{N}$ . The bra's and ket's form an orthogonal basis of  $\mathcal{A} = \mathbb{C}^{\mathbb{N}}$  with inner product  $\langle k | k' \rangle = \delta_{k,k'}$ . Relations (9), (14) then provide a representation of the matrices  $A_a$ .

In order to satisfy the boundary conditions involving the quantum boundary fields and the Lindblad dissipators we further define  $\Phi = [g, \Omega]$  and the tensor products  $\Xi_k = \Omega^{\otimes k-1} \otimes \Xi \otimes \Omega^{\otimes N-k}$ ,  $\Phi_1 = \Phi \otimes \Omega^{\otimes N-1}$ ,  $\Phi_N = \Omega^{\otimes N-1} \otimes \Phi$ .

The local divergence condition implies  $[H, \Omega^{\otimes N}] = \Phi_1^L + \Xi_1 + \Phi_N^R - \Xi_N$ . The stationary Lindblad equation (5) can thus be split into two equations

$$\mathcal{D}^L(SS^\dagger) = i(\Phi_1^L + \Xi_1)S^\dagger - iS(\Phi_1^{L\dagger} + \Xi_1^\dagger) \quad (17)$$

$$\mathcal{D}^R(SS^\dagger) = i(\Phi_N^R - \Xi_N)S^\dagger - iS(\Phi_N^{R\dagger} - \Xi_N^\dagger) \quad (18)$$

for each boundary. Using the decomposition  $S = \langle \phi | [\mathbb{1}A_0 + \sigma^z A_z + A_+ \sigma^+ + A_- \sigma^-] \otimes \Omega^{\otimes N-1} | \psi \rangle$  for the first equation and  $S = \langle \phi | \Omega^{\otimes N-1} \otimes [A_0 \mathbb{1} + A_z \sigma^z + A_+ \sigma^+ + A_- \sigma^-] | \psi \rangle$  for the second equation and factoring out the term containing  $\Omega^{\otimes N-1}$  yields two sets of equations for the action of the matrices  $A_a$  on the vectors  $\langle \phi |$  and  $| \psi \rangle$  respectively. In this letter we outline this programme for the isotropic chain  $\Delta = 1$ . The construction for  $\Delta \neq 1$  is conceptually similar, but technically more involved and will be presented in a more detailed paper [18].

For taking the isotropic limit  $q \rightarrow 1$  we choose the normalization factors  $\alpha = \lambda = 1$  and set  $\mu = \nu$  in (12), (13) and arrive at

$$\Omega = \begin{pmatrix} \nu S_z & i S_+ \\ i S_- & \nu^{-1} S_z \end{pmatrix}, \quad \Xi = \begin{pmatrix} \nu & 0 \\ 0 & -\nu^{-1} \end{pmatrix}. \quad (19)$$

The quadratic relations for the quantum algebra turn into the usual commutation relations  $[S_+, S_-] = 2S^z$ ,  $[S_z, S_\pm] = \pm S_\pm$  for  $SU(2)$ . The irreducible representation (17) turns into an irrep of  $SU(2)$  by observing that  $[x]_1 = x$ . Since the Lindblad dissipators do not generate terms proportional to the unit matrix, we cancel these terms that appear on the r.h.s. of (5) by setting  $\nu = i$  which leads to  $A_0 = 0$  and  $\Omega = i\vec{S} \cdot \vec{\sigma}$ , where  $\vec{S} = (\frac{S_+ + S_-}{2}, i\frac{S_+ - S_-}{2}, S_z)$  and  $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ , or in terms of the  $\sigma^\pm$  the form  $\Omega = i(S_z \sigma^z + S_+ \sigma^+ + S_- \sigma^-)$ .

The key step in solving the boundary equations is the introduction of coherent states

$$\langle \phi | := \sum_{n=0}^{\infty} \frac{\phi^n}{n!} \langle 0 | (S_+)^n = \sum_{n=0}^{\infty} \phi^n \langle n | \quad (20)$$

$$| \psi \rangle := \sum_{n=0}^{\infty} \frac{\psi^n}{n!} (S_-)^n | 0 \rangle = \sum_{n=0}^{\infty} \psi^n \binom{2p}{n} | n \rangle. \quad (21)$$

Using the commutation relations of  $SU(2)$  one finds

$$\langle \phi | S_z = \langle \phi | (p - \phi S_+), \quad \langle \phi | S_- = \phi \langle \phi | (2p - \phi S_+) \quad (22)$$

and

$$S_z | \psi \rangle = (p - \psi S_-) | \psi \rangle, \quad S_+ | \psi \rangle = \psi (2p - \psi S_-) | \psi \rangle \quad (23)$$

The left boundary equations can now be solved by noting that the Lindblad operator (3) can be obtained from a complete polarization along the  $z$ -axis by the unitary transformation  $U = e^{i\frac{\theta_L}{2}\sigma^x}$  on site 1 of the chain which rotates the  $z$ -axis into a new direction  $u$ . After this transformation the leftmost matrix  $\Omega$  in the tensor product  $\Omega^{\otimes N}$  reads in the new basis

$$\Omega(\theta_L) = i(S_z(\theta_L)\sigma_u^z + S_+(\theta_L)\sigma_u^+ + S_-(\theta_L)\sigma_u^-) \quad (24)$$

with the new components

$$\begin{aligned} S_z(\theta_L) &= S_z \cos \theta_L + i \sin \theta_L \frac{S_+ - S_-}{2} \\ S_+(\theta_L) &= \frac{S_+ + S_-}{2} + \cos \theta_L \frac{S_+ - S_-}{2} + i S_z \sin \theta_L \\ S_-(\theta_L) &= \frac{S_+ + S_-}{2} - \cos \theta_L \frac{S_+ - S_-}{2} - i S_z \sin \theta_L. \end{aligned} \quad (25)$$

In order to solve the left boundary equation we need to impose

$$\langle \phi | S_-(\theta_L) = 0, \quad \langle \phi | S_z(\theta_L) = p \langle \phi | \quad (26)$$

Using (22) these conditions are satisfied if the coherent state parameter  $\phi$  is chosen to be

$$\phi = i \tan(\theta_L/2). \quad (27)$$

In order to prove this result we point out (22), (23) can be used to express vectors of the form  $\langle \phi | (a + bS_z + cS_+ + dS_-)$  that appear in the boundary equations just in terms of e.g.  $\langle \phi | (a' + d'S_-)$ , and similarly for the action on ket-vectors  $| \psi \rangle$ . The choice (27) leads to

$$S = \langle \phi | \Omega^{\otimes N} | \psi \rangle = ip\sigma_u^z \otimes \tilde{S} + \sigma_u^+ \otimes W \quad (28)$$

where  $\tilde{S} = \langle \phi | \Omega^{\otimes N-1} | \psi \rangle$  and  $W = i \langle \phi | (S_+ + S_-) \Omega^{\otimes N-1} | \psi \rangle$ . Moreover,

$$SS^\dagger = |p|^2 \mathbb{1} \otimes \tilde{S}S^\dagger - ip\sigma_u^- \otimes \tilde{S}W^\dagger - (ip)^* \sigma_u^+ \otimes W\tilde{S}^\dagger + \sigma_u^+ \sigma_u^- \otimes WW^\dagger. \quad (29)$$

Now, on the one hand we see that the action of the left dissipator  $\mathcal{D}^L$  leads to

$$\mathcal{D}^L(SS^\dagger) = 2\Gamma|p|^2 \sigma_u^z \otimes \tilde{S}\tilde{S}^\dagger + \Gamma ip\sigma_u^- \otimes \tilde{S}W^\dagger + \Gamma(ip)^* \sigma_u^+ \otimes W\tilde{S}^\dagger. \quad (30)$$

On the other hand the left contribution of the unitary part of the Lindblad equation leads to

$$\begin{aligned} i[H, SS^\dagger]_{Left} &= -(ip + (ip)^*) \sigma_u^z \otimes \tilde{S}\tilde{S}^\dagger \\ &\quad - \sigma_u^- \otimes (1 - 2if^L(ip)) \tilde{S}W^\dagger \\ &\quad - \sigma_u^+ \otimes (1 + 2if^L(ip)^*) W\tilde{S}^\dagger \end{aligned} \quad (31)$$

Comparing the two contributions gives the solution for the representation parameter

$$p = \frac{i}{\Gamma - 2if^L}. \quad (32)$$

The right boundary is treated along the same lines. The right-end state is polarized in the  $(x, z)$  plane in a direction  $v$  generated by the rotation

$$U = e^{i\frac{\theta_R}{2}\sigma^y}, \quad (33)$$

where we take as reference the  $-z$ -direction. Going through similar steps as above we impose the cancellation

$$S_+(\theta_R)|\psi\rangle = 0. \quad (34)$$

With (23) this yields to

$$\psi = -\tan(\theta_R/2). \quad (35)$$

In order to fulfill the stationarity condition (18), together with (23) one needs to impose  $f^R = -f^L$  such that the representation parameter takes the value given in (32). Interestingly, this condition turns out to allow for the inclusion of a Dzyaloshinsky-Moriya interaction in the XXZ-Hamiltonian [18] which is the key ingredient in the Lagrange-multiplier approach of [21] to current-carrying states of quantum spin systems.

In conclusion, the solution of the completely polarized twisted case with a polarization on the left in the  $(y, z)$  plane and in the right in the  $(x, z)$  plane is given by the Matrix Product Ansatz for  $S$  in the form (6) with coherent state parameters (27), (35) and representation parameter (32). At  $\theta_L = \theta_R = 0$  and vanishing boundary fields  $f^R = f^L = 0$  one recovers the untwisted solution [12].

The model with a twist is fundamentally different from the untwisted one, which can be seen by studying one- and two-point functions in the steady state. Note that in the isotropic model, all three spin projections  $\sigma_n^x$ ,  $\sigma_n^y$  and  $\sigma_n^z$  are locally conserved, i.e.,  $\frac{d}{dt}\sigma_n^\alpha = \hat{j}_{n-1,n}^\alpha - \hat{j}_{n,n+1}^\alpha$ , where  $\hat{j}_{n,n+1}^\alpha = 2 \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma} \sigma_n^\beta \sigma_{n+1}^\gamma$  ( $\varepsilon_{\alpha\beta\gamma}$  being Levi-Civita symbol). This leads to three different steady state currents  $j^\alpha = \langle \hat{j}_{n,n+1}^\alpha \rangle$  for  $\alpha = x, y, z$ . In the untwisted model ( $\theta_L = \theta_R = 0$ ) two out of three one-point correlations vanish in the steady state,  $\langle \sigma_n^x \rangle = \langle \sigma_n^y \rangle = 0$  for all  $n$ , corresponding to trivial flat  $x$ - and  $y$ - magnetization profiles along the chain. Also, two out of three spin currents are completely suppressed in the untwisted setup  $j^x = j^y = 0$ . In a model with a twist, *neither* of the one-point functions vanishes, and *all three* spin currents  $j^x, j^y, j^z$  are generically nonzero.

In order to see this, we note that in the usual untwisted model with Lindblad operators being creation/annihilation operators [11, 12] the steady state is invariant under a parity sym-

metry  $\rho = U\rho U$  where  $U = (\sigma^z)^{\otimes N} = U^{-1}$  [19]. Any physical observable that changes sign under the parity operation has to vanish in the steady state, e.g.  $\langle \sigma_n^x \rangle = \text{Tr}(U\sigma_n^x U\rho) = -\text{Tr}(\sigma_n^x \rho) = -\langle \sigma_n^x \rangle$ , from which  $\langle \sigma_n^x \rangle = 0$  follows. In this way one readily obtains  $\langle \sigma_n^x \rangle = \langle \sigma_n^y \rangle = j^x = j^y = 0$ .

In the isotropic model with a twist the parity symmetry is broken, but its place is taken by another symmetry, which we specify here for twisting angles  $\theta_R = -\theta_L = \pi/2$ : It involves left-right reflection  $R(A \otimes B \otimes \dots \otimes C) = (C \otimes \dots \otimes B \otimes A)R$ , global rotation in the  $XY$ -plane  $U_{\text{rot}} = \text{diag}(1, i)^{\otimes N}$  and  $\Sigma_x = (\sigma^x)^{\otimes N}$  and reads  $\rho = V\rho V^\dagger$ , where  $V = \Sigma_x U_{\text{rot}} R$  [20]. It is straightforward to check that neither of the set of observables  $\langle \sigma_n^\alpha \rangle, j^\alpha$ , changes sign under the  $V$  symmetry, and therefore they are generically nonzero. The symmetry  $V$  does, however, give rise to nontrivial relations between the observables, e.g.  $j^x = -j^y$ ,  $\langle \sigma_n^z \rangle = -\langle \sigma_{N-n}^z \rangle$ , etc..

The major novelty of our approach is the exact MPA solution by a quadratic algebra which turns out to be the symmetry algebra of the unitary evolution of the bulk part of the Hamiltonian. Remarkably, this MPA solves the stationary Lindblad equation even though both the quantum boundary fields and the Lindblad dissipators destroy this symmetry. We expect that Lindblad equations for other open boundary-driven many-body quantum systems with a  $q$ -deformed non-Abelian bulk symmetry can be solved in a similar fashion. Since representations of the corresponding quantum algebras are known, exact results for observables become available. An open problem is the relationship of the MPA to the integrability of the bulk Hamiltonian and hence to the extension of the MPA approach to dynamical observables. Work on eigenfunctions [22, 23] and recent exact results by Eisler for the density matrix with bulk Lindblad terms [24] hint at this possibility.

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